SHIGA UNIVERSITY

CRR WORKING PAPER SERIES B

Working Paper No. B-9

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A Robust Recursive Utility under Jump-Diffusion Information

by

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July 2006

Abstract. This paper generalizes the utility with a robust control criterion, developed by Hansen, Sargent, and Tallarini [14], Anderson, Hansen, and Sargent [2], and Skiadas [27], to a recursive one under non-Markovian jump-diffusion information, and presents the existence and uniqueness of the generalized utility under some regualrity condition by utilizing approaches of Skiadas [27] and Schroder and Skiadas [25]. This paper also shows that the generalized utility admits a normalized representation, where future utility enters the recursion through an aggregator, and examines the generalized utility's basic properties including "ambiguity aversions." JEL No. C62, D81, D91.

This paper is a revision and expansion of my working paper (Kusuda [20]). I would like to thank Professor Jan Werner for his invaluable advice. I am grateful for comments from participants in the presentation at University of Minnesota. This research was supported in part by a grant from the Ministry of Education, Sport, Science and Technology, Japan.

1. Introduction

Utilities with "Knightian uncertainty" or "ambiguity" have been studied for the past decade.[1] The utility with a stochastic robust control criterion, developed by Hansen, Sargent, and Tallarini [14], Anderson, Hansen, and Sargent [2], Lazrak and Quenez [21], and Skiadas [27], is one of such utilities. This paper tentatively calls the utility the *robust utility*. The standard robust utility takes the form of minimization, over a set of probabilities equivalent to a reference probability, of a time-additive utility plus the *relative entropy* which penalizes a deviation from the reference probability. It, therefore, must be proven that the utility functional is well-defined, that is, the utility functional uniquely exists. Anderson, Hansen, and Sargent [2] presented the standard robust utility's existence, uniqueness, and a representation with a semigroup generator under Markovian jump-diffusion information. Lazrak and Quenez [21] showed that the standard robust utility's existence, uniqueness, and a stochastic differential utility (SDU, hereafter) representation under non-Markovian diffusion information. Skiadas [27] generalized the underlying time-additive utility of the standard robust utility to a non-recursive non-time-additive utility including as special cases, habit formation utilities[2] and the Hindy-Huang-Kreps utility[3], and presented the existence, uniqueness, and *normalized SDU representation*, where future utility enters the recursion through an aggregator, of the robust utility under non-Markovian diffusion information. Skiadas [27] also gave an outline of the proof for similar results in the case when the underlying utility is SDU. The purpose of this paper is to generalize the underlying utility of robust utility to a recursive one including as special cases, the SDU, recursive versions of habit formation utilities, and a recursive version of Hindy-Huang-Kreps utility[4], under non-Markovian jump-diffusion information, and then, to examine the generalized utility's existence, uniqueness, representation, and basic properties including time consistency, preferences for information, and "ambiguity aversions."

This paper is summarized as follows. A *dynamic utility* (Duffie and Skiadas [10]), which is a recursive utility including as special cases, the SDU, the recursive habit formation utility, and the recursive Hindy-Huang-Kreps utility, is first introduced under a non-Markovian jump-diffusion information. Then the stochastic robust control criterion advocated by Anderson, Hansen, and Sargent [2], is applied to the dynamic utility to introduce ambiguity aversion. This paper calls the utility the *robust dynamic utility* (RDU, hereafter). First, it is presented by employing the same approach as Skiadas [27] that under some regularity condition, for each consumption process, the utility process of RDU is a unique solution to a backward

[1]See, *e.g.*, Anderson, Hansen, and Sargent [2], Chen and Epstein [5], Epstein [11], Epstein and Zhang [13], Hansen, Sargent, and Tallarini [14], Hansen, Sargent, Turmuhambetova, and Williams [15], Lazrak and Quenez [21], Maccheroni, Marinacci, and Rustichini [22], Skiadas [27], and Wang [29].

[2]Habit formation utilities are divided into two types, *i.e.*, the internal habit formation utility and the external habit formation utility. The internal habit formation utility was introduced by Ryder and Heal [24], and has been developed by Sundaresan [28], Constantinides [6], Detemple and Zapareto [8], and Dai [7]. The external habit formation utility was proposed by Abel [1], and extended by Campbell and Cochrane [4].

[3]The Hindy-Huang-Kreps utility was introduced by Hindy, Huang, and Kreps [17], and generalized by Hindy and Huang [16].

[4]Recursive versions of internal habit formation utility and of Hindy-Huang-Kreps utility were introduced by Duffie and Epstein [9] and extended by Duffie and Skiadas [10].

stochastic differential-difference equation (BSDDE, hereafter), if there exists the unique solution to the BSDDE. Then the existence and uniqueness of solutions to the BSDDE is proven by generalizing the approach of Schroder and Skiadas [25] for an SDU under diffusion information. The RDU process also admits an unnormalized SDU-like representation under jump-diffusion information, but this unnormalized representation of RDU is not so analytically tractable. While any SDU is normalized under pure diffusion information, this does not apply to the case of jump-diffusion information. Recently, a sufficient condition for the normalization of SDU has been revealed under jump-diffusion information (Kusuda [19]). It is confirmed that this condition holds for the RDU, and therefore the normalized RDU is obtained. Next, this paper exmines the RDU's basic properties including time consistency, preferences for information, and monotonicity by generalizing results of Duffie and Epstein [9] and of Skiadas [26] for normalized SDUs. Finally, attitudes toward ambiguity of agents with RDU are examined by mainly adopting the notions proposed by Chen and Epstein [5].

The remainder of this paper is organized into four sections and four appendices. Section 2 introduces the RDU under jump-diffusion information. Section 3 presents the existence and uniqueness of RDU. Section 4 reveals that the RDU admits a normalized representation. Section 5 examines the RDU's basic properties. Appendix A, B, and C introduce marked point process, Ito's Formula and Girsanov's Theorem, and an extension of Gronwall-Bellman Inequality, respectively. Appendix D shows proofs of lemmas.

2. Robust Dynamic Utility under Jump-Diffusion Information

This section first reviews the dynamic utility and then introduces the RDU under jump-diffusion information.

A continuous-time model with finite time span $\mathbf{T} := [0, T]$ and with a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is assumed, where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is the natural filtration generated by a d-dimensional Wiener process W and a *marked point process* $\nu(dt \times dz)$ on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ (in usual applications, $\mathbb{Z} = \mathbb{R}^{d'}$, or $\mathbb{N}^{d'}$, or a finite set) which is independent of W and has the P-intensity kernel $\lambda_t(dz)$ (for marked point process, see Appendix A). A convex set $\mathcal{C}$ of semimartingales is taken as primitive. Any element C of $\mathcal{C}$ represents some cumulative consumtpion process, meaning that for every time $t \in \mathbf{T}$, C_t represents the total net consumption up to time t. Let $\mathbf{L}^\infty := \mathbf{L}^\infty(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where $\mathcal{P}$ is the predictable σ-algebra, and μ is the product measure of P and the Lebesgue measure on $\mathbf{T}$. For each $n \in \mathbb{N}$, the space of $\mathcal{P}$-measurable real-valued processes Y satisfying the integrability condition $\int_0^T |Y_s|^n\, ds < \infty$ P-a.s., is denoted by $\mathcal{L}^n$. The space of $\mathcal{P} \otimes \mathcal{Z}$-measurable real-valued process H satisfying the integrability condition $\int_0^T\int_{\mathbb{Z}} |H_s(z)|\, \lambda_s(dz)\, ds < \infty$ P-a.s., is denoted by $\mathcal{L}(\lambda_t(dz) \times dt)$. The expectation operator under P is denoted by E, and the conditional expectation operator under P given $\mathcal{F}_t$ is denoted by E_t.

2.1. Dynamic Utility.

First, the dynamic utility is reviwed following Duffie and Skiadas [10]. Let $X : \mathcal{C} \to \mathbf{L} := \prod_{j=1}^{n_0} \mathbf{L}^\infty(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where $\prod$ denotes Cartesian product. Let $f : \Omega \times \mathbf{T} \times \mathbb{R}^{n_0} \times \mathbb{R} \to \mathbb{R}$ denote a function such that $f(\,\cdot\,, \cdot\,, x, u)$ is a predictable process for every (x, u). Let $\beta \in \mathbb{R}_+$ and $f^\beta(x, u) = f(x, u) - \beta u$. A functional $U : \mathcal{C} \to \mathbb{R}$ is said to be a *dynamic utility with characteristic* (f, X, β) or (f^β, X) if and only if $U(C) = U_0(C)$ for every $C \in \mathcal{C}$ where $U_0(C)$ is the initial value of unique solution $U_t(C)$ in $\mathbf{L}^\infty$ to the recursion

$$U_t(C) = E_t\left[\int_t^T e^{-\beta(s-t)} f(X_s(C), U_s(C))\, ds\right] \qquad \forall t \in \mathbf{T}, \tag{2.1}$$

or equivalently

$$U_t(C) = E_t\left[\int_t^T f^\beta(X_s(C), U_s(C))\, ds\right] \qquad \forall t \in \mathbf{T}. \tag{2.2}$$

The pair (f^β, X) or the function f^β is called the *aggregator* of the dynamic utility U.

The dynamic utility is a generalization of SDU, and was first introduced by Duffie and Skiadas [10]. The dynamic utility includes as special cases, the SDU, the recursive internal and external habit formation utilities, and the recursive Hindy-Huang-Kreps utility as shown below.

Example 1. Suppose $C_t = \int_0^t c_s ds$ and $X_t(C) = c_t$. Then U is the SDU introduced by Duffie and Epstein [9]. In particular, if $f(c_t, U_t) = v(c_t)$ where v is a von Neumann-Morgenstern utility function, then U is a standard time-additive utility.

Example 2. Suppose $C_t = \int_0^t c_s ds$ and $X_t(C) = (c_t, x_t)$ where $x_t = x_0 + \int_0^t h^0(c_s, x_s)\, ds + \int_0^t h(c_s, x_s) \cdot dW_s + \int_0^t \int_{\mathbb{Z}} H(c_s, x_s, z)\nu(ds \times dz)$ for some $h^0 : \mathbf{T} \times \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbf{T} \times \mathbb{R}^2 \to \mathbb{R}^d$, and $H : \mathbf{T} \times \mathbb{R}^2 \times \mathbb{Z} \to \mathbb{R}$ such that h^0, h, and $\int_{\mathbb{Z}} H(c_s, x_s, z)\nu(ds \times dz)$. If $f(x, u) = v(x)$ and $H = 0$ then U is a stochastic internal habit formation utility which was introduced by Ryder and Heal [24], and has been extended by Sundaresan [28], Constantinides [6], Detemple and Zapareto [8], and Dai [7]. If f and H are general, then U is a recursive stochastic internal habit formation utility, which is a generalization of the recursive stochastic internal habit formation utility developed by Duffie and Epstein [9] and Duffie and Skiadas [10].

Example 3. Suppose $C_t = \int_0^t c_s ds$ and $X_t(C) = x_0 + \int_0^t h^0(c^A_s, X_s)\, ds + \int_0^t h(c^A_s, X_s) \cdot dW_s + \int_0^t \int_{\mathbb{Z}} H(c^A_s, X_s, z)\nu(ds \times dz)$ where $c^A \in \mathbf{L}^\infty$ is aggregate consumption process, $h^0 : \mathbf{T} \times \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbf{T} \times \mathbb{R}^2 \to \mathbb{R}^d$, and $H : \mathbf{T} \times \mathbb{R}^2 \times \mathbb{Z} \to \mathbb{R}$ such that h^0, h, and H satisfy the same conditions as the above. If $f(x, u) = v(x)$, $h = 0$, and $H = 0$, then U is the external habit formation utility proposed by Abel [1], and extended by Campbell and Cochrane [4]. If f, h, and H are general, then U is a recursive stochastic external habit formation utility.

Example 4. Suppose $X_t(C) = x_0 + \int_0^t h_s\, dC_s$ for some $h^0 : \Omega \times \mathbf{T} \to \mathbb{R}$ such that h is a bounded predictable process. If $f(x, u) = v(x)$, then U is a Hindy-Huang-Kreps utility which was introduced by Hindy, Huang, and Kreps [17] and generalized by Hindy and Huang [16]. If f is general, then U is a recursive Hindy-Huang-Kreps utility introduced by Duffie and Epstein [9] and extended by Duffie and Skiadas [10].

A sufficient condition for the existence and uniqueness of the dynamic utility $U(C)$ given by Duffie and Epstein [9] and Duffie and Skiadas [10], is as follows: The aggregator f^β of dynamic utility satisfies the following conditions:

Uniform Lipschitz condition in utility: There exists $k_1 \in \mathbb{R}_{++}$ such that

$$|f^\beta(x, u) - f^\beta(x, \tilde{u})| \leqslant k_1\, |u - \tilde{u}| \qquad \forall(\omega, t, x, u, \tilde{u}).$$

Growth condition in consumption: There exists $k_2 \in \mathbb{R}_{++}$ such that

$$|f^\beta(x,0)| \leqslant k_2(1+\|x\|) \qquad \forall(\omega,t,x).$$

where $\|\cdot\|$ is Euclidean norm.

Let $\mathbf{B}_{X(\mathcal{C})}$ denote the space of functions $g:\Omega\times\mathbf{T}\times\mathbb{R}^{n_0}\to\mathbb{R}_{++}$ such that for each x, $g(\,\cdot\,,\,\cdot\,,x)$ is a predictable process, and for each $C\in\mathcal{C}$, $\operatorname{esssup}_{(\omega,t)\in\Omega\times\mathbf{T}} g(\omega,t,X_t(C))<K$ for some $K\in\mathbb{R}_{++}$. The above condition is slightly relaxed as follows.

Assumption 1. *The function f^β satisfies the following conditions:*

Lipschitz condition in utility: *There exists a function $k_1\in\mathbf{B}_{X(\mathcal{C})}$ such that*

$$|f^\beta(x,u)-f^\beta(x,\tilde{u})| \leqslant k_1(\omega,t,x)\,|u-\tilde{u}| \qquad \forall(\omega,t,x,u,\tilde{u}).$$

Generalized growth condition in consumption: *There exists a function $k_2\in\mathbf{B}_{X(\mathcal{C})}$ such that*

$$|f^\beta(x,0)| \leqslant k_2(\omega,t,x) \qquad \forall(\omega,t,x).$$

Conditions on f^β in Assumption 1 can be also interpreted as restrictions on the consumption space C. The following proposition is an extension of the results of Duffie and Epstein [9] and Duffie and Skiadas [10].

Proposition 1. *Under Assumption 1, for every $C\in\mathcal{C}$, there exists a unique solution $U(C)\in\mathbf{L}^\infty$ to recursion (2.1).*

Proof. Let $C\in\mathcal{C}$. Define a mapping $F:\mathbf{L}^\infty\to\mathbf{L}^\infty$ by

$$F_t(U)=E_t\left[\int_t^T f^\beta(X_s(C),U_s)\,ds\right] \qquad \forall t\in\mathbf{T}. \tag{2.3}$$

It follows from Assumption 1, $X(C)\in\mathbf{L}^\infty$, and $(k_1,k_2)\in\mathbf{B}_{X(\mathcal{C})}\times\mathbf{B}_{X(\mathcal{C})}$ that for each $t\in\mathbf{T}$,

$$\begin{aligned}|F_t(U)| &\leqslant E_t\left[\int_t^T\Big(|f^\beta(X_s(C),U_s)-f^\beta(X_s(C),0)|+|f^\beta(X_s(C),0)|\Big)\,ds\right]\\ &\leqslant \int_t^T\Big(\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} k_1(X_s(C))\,|U_s|+\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} k_2(X_s(C))\Big)\,ds\\ &\leqslant K_1(T-t)\|U\|_{\mathbf{L}^\infty}+K_2(T-t)\end{aligned}$$

for some $(K_1,K_2)\in\mathbb{R}^2_{++}$. Thus, $F(U)\in\mathbf{L}^\infty$, and $F:\mathbf{L}^\infty\to\mathbf{L}^\infty$ is well-defined. Let $(U,\tilde{U})\in\mathbf{L}^\infty\times\mathbf{L}^\infty$ and $t\in\mathbf{T}$. The following holds:

$$\begin{aligned}|F_t(U)-F_t(\tilde{U})| &\leqslant E_t\left[\int_t^T|f^\beta(X_s(C),U_s)-f^\beta(X_s(C),\tilde{U}_s)|\,ds\right]\\ &\leqslant E_t\left[\int_t^T k_1(X_s(C))\,|U_s-\tilde{U}_s|\,ds\right]\\ &\leqslant KE_t\left[\int_t^T|U_s-\tilde{U}_s|\,ds\right]\leqslant K(T-t)\|U-\tilde{U}\|_{\mathbf{L}^\infty}\end{aligned} \tag{2.4}$$

where $K = \operatorname{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} k_1(X_s(C))$. It follows from (2.4) that

$$|F_t^{(2)}(U) - F_t^{(2)}(\tilde{U})| \leqslant E_t\left[\int_t^T K|F_s(U) - F_s(\tilde{U})|\,ds\right]$$
$$\leqslant K^2\|U-\tilde{U}\|_{\mathbf{L}^\infty}\int_t^T (T-s)\,ds \leqslant \frac{\{K(T-t)\}^2}{2!}\|U-\tilde{U}\|_{\mathbf{L}^\infty}.$$

Repeating this calculation yields

$$|F_t^{(n)}(U) - F_t^{(n)}(\tilde{U})| \leqslant \frac{\{K(T-t)\}^n}{n!}\|U-\tilde{U}\|_{\mathbf{L}^\infty} \qquad \forall n\in\mathbb{N},$$

and therefore,

$$\|F^{(n)}(U) - F^{(n)}(\tilde{U})\|_{\mathbf{L}^\infty} \leqslant \frac{(KT)^n}{n!}\|U-\tilde{U}\|_{\mathbf{L}^\infty} \qquad \forall n\in\mathbb{N}.$$

Thus, for n large enough, $F^{(n)}$ is a contraction mapping. Hence, there exists a unique fixed point $U^C \in \mathbf{L}^\infty$, *i.e.*, $F^{(n)}(U^C) = U^C$. Since $F^{(n)}(F(U^C)) = F(F^{(n)}(U^C)) = F(U^C)$, it follows by uniqueness that $F(U^C) = U^C$. The uniqueness of fixed points of F follows from that of $F^{(n)}$. Therefore, $U(C) := U^C$ is the unique solution in $\mathbf{L}^\infty$ to the recursion (2.1). □

2.2. **Robust Dynamic Utility.** Next, the RDU is introuduced. Let $\mathbb{P}$ be the set of all probability measures on $(\Omega,\mathcal{F})$ that are *equivalent to* P, *i.e.* they define the same null events as P. It follows from *Girsanov's Theorem* (see Appendix B.2) that an equivalent measure $P^{v,w}$ is characterized by the Radon-Nikodym derivative $dP^{v,w} = \Lambda_T^{v,w}\,dP$ where $\Lambda_T^{v,w}$ is given by

$$\Lambda_T^{v,w} = \exp\biggl(\int_0^T\Bigl(-\frac{1}{2}\|v_t\|^2 - \int_{\mathbb{Z}} w_t(z)\,\lambda_t(dz)\Bigr)dt + \int_0^T v_t\cdot dW_t + \int_0^T \log(1+w_t(z))\,\nu(dt\times dz)\biggr) \tag{2.5}$$

where $v_t \in \prod_{j=1}^d \mathcal{L}^2$, $w_t(z)\in\mathcal{L}(\lambda_t(dz)\times dt)$ satisfying $w_t(z) > -1$ P-a.s. for every $(t,z)\in\mathbf{T}\times\mathbf{Z}$, and $v_t\cdot dW_t = \sum_{j=1}^d v_{jt}dW_{jt}$. The expectation operator under $P^{v,w}$ is denoted by $E^{v,w}$, and the conditional expectation operator under $P^{v,w}$ given $\mathcal{F}_t$ is denoted by $E_t^{v,w}$. An RDU with the reference probability P takes the form of minimization, over s set of equivalent probability measures, of a dynamic utility plus the *discounted relative entropy of* $P^{v,w}$ *with respect to* P, which penalizes a deviation from the reference probability P. Here the *discounted relative entropy of* $P^{v,w}$ *with respect to* P is defined by

$$\tilde{\mathcal{R}}_t^{v,w} = E_t^{v,w}\left[\beta\int_t^T e^{-\beta(s-t)}\log\left(\frac{\Lambda_s^{v,w}}{\Lambda_t^{v,w}}\right)ds + e^{-\beta(T-t)}\log\left(\frac{\Lambda_T^{v,w}}{\Lambda_t^{v,w}}\right)\right] \qquad \forall t\in\mathbf{T}. \tag{2.6}$$

Let $\zeta\in\mathbb{R}_{++}$. For each $C\in\mathcal{C}$, define a process $U^{v,w}(C)$ in $\mathbf{L}^\infty$ by

$$U_t^{v,w}(C) = E_t^{v,w}\left[\int_t^T e^{-\beta(s-t)} f(X_s(C), U_s^{v,w}(C))\,ds\right] + \frac{1}{\zeta}\tilde{\mathcal{R}}_t^{v,w} \quad \forall t\in\mathbf{T}, \tag{2.7}$$

or equivalently

$$U_t^{v,w}(C) = E_t^{v,w}\left[\int_t^T f^\beta(X_s(C), U_s^{v,w}(C))\,ds\right] + \frac{1}{\zeta}\mathcal{R}_t^{v,w} \quad \forall t\in\mathbf{T} \tag{2.8}$$

where $\mathcal{R}_t^{v,w}$ is the *relative entropy of* $P^{v,w}$ *with respect to* P, defined by

$$\mathcal{R}_t^{v,w} = E_t^{v,w}\left[\log\left(\frac{\Lambda_T^{v,w}}{\Lambda_t^{v,w}}\right)\right].$$

Let $\bar{\mathbb{P}}$ denote the set of all $P^{v,w} \in \mathbb{P}$ such that $U^{v,w}(C)$ is well-defined for every $C \in \mathcal{C}$. The notion of *RDU* is defined in the following.

Definition 1. A functional $\hat{U} : \mathcal{C} \to \mathbb{R}$ is said to be an *RDU with characteristic* (f, X, β, ζ) or (f^β, X, ζ) if and only if for every $C \in \mathcal{C}$, $\hat{U}(C)$ satisfies

$$\hat{U}(C) = \min_{P^{v,w} \in \bar{\mathbb{P}}} U_0^{v,w}(C). \tag{2.9}$$

Remark 1. For $\zeta = 0+$, the minimizer of (2.9) is the reference probability P. As ζ increases, the relative entropy of the minimizer of (2.9) with respect to the reference probability P becomes larger. It shall be shown in Section 5 that the larger the parameter ζ is, the more "ambiguity averse" the agent is. For an axiomatic treatment of robust utility in atemporal setting, refer to Maccheroni, Marinacci, and Rustichini [22], and Wang [29].

3. Existence and Uniqueness of RDU

This section presents that under Assumption 1, the RDU defined by Definition 1 is well-defined, and the utility process of RDU is a unique solution to a BSDDE (Backward Stochastic Differential-Difference Equation). First, it is presented by employing the same approach as Skiadas [27] that under Assumption 1, for each consumption process, the utility process of RDU is a unique solution to a BSDDE, if there exists the unique solution to the BSDDE. Then the existence and uniqueness of solutions to the BSDDE is proven by generalizing the approach of Schroder and Skiadas [25] for an SDU under diffusion information.

3.1. **An Expression for Discounted Relative Entropy.** The discounted relative entropy $\tilde{\mathcal{R}}_t^{v,w}$ defined by (2.6) is expressed as shown in Lemma 1, which extends the results for Markovian jump-diffusion information (Anderson, Hansen, and Sargent [2]) and for non-Markovian diffusion information (Lazrak and Quenez [21], and Skiadas [27]).

Lemma 1. *Let* $P^{v,w} \in \bar{\mathbb{P}}$. *For every* $t \in \mathbf{T}$, $\tilde{\mathcal{R}}_t^{v,w}$ *is expressed as*

$$\tilde{\mathcal{R}}_t^{v,w} = E_t^{v,w}\left[\int_t^T e^{-\beta(s-t)}\left\{\frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}}\left(\frac{w_s(z)}{1+w_s(z)} + \log(1+w_s(z))\right)\lambda_s^{v,w}(dz)\right\} ds\right]. \tag{3.1}$$

Proof. Applying integration by parts to (2.6) yields

$$\tilde{\mathcal{R}}_t^{v,w} = E_t^{v,w}\left[\int_t^T e^{-\beta(s-t)} d\left(\log\frac{\Lambda_s^{v,w}}{\Lambda_t^{v,w}}\right)\right] \qquad \forall t \in \mathbf{T}. \tag{3.2}$$

It follows from (2.5) that

$$\begin{aligned}\log\frac{\Lambda_s^{v,w}}{\Lambda_t^{v,w}} &= \int_t^s \left\{-\frac{1}{2}\|v_{s'}\|^2 - \int_{\mathbb{Z}} w_{s'}(z)\,\lambda_{s'}(dz)\right\} ds' \\ &\quad + \int_t^s v_{s'} \cdot dW_{s'} + \int_t^s \log(1+w_{s'}(z))\,\nu(ds' \times dz) \qquad \forall s \in [t,T] \end{aligned} \tag{3.3}$$

for every $t \in \mathbf{T}$. It follows from Girsanov's Theorem that a $P^{v,w}$-Wiener process $W^{v,w}$ and the $P^{v,w}$-intensity kernel $\lambda^{v,w}$ of ν satisfy

$$dW_t = dW_t^{v,w} + v_t\,dt \qquad \forall t \in \mathbf{T}, \tag{3.4a}$$

$$\lambda_t(dz) = \frac{\lambda_t^{v,w}(dz)}{1+w_t(z)} \qquad \forall (t,z) \in \mathbf{T}\times\mathbb{Z}, \tag{3.4b}$$

respectively. Substituting (3.4a) and (3.4b) into (3.3) yields

$$\begin{aligned}\log\frac{\Lambda_s^{v,w}}{\Lambda_t^{v,w}} &= \int_t^s \left\{\frac{1}{2}\|v_{s'}\|^2 + \int_{\mathbb{Z}}\left(\frac{-w_{s'}(z)}{1+w_{s'}(z)} + \log(1+w_{s'}(z))\right)\lambda_{s'}^{v,w}(dz)\right\} ds' \\ &\quad + \int_t^s v_{s'}\cdot dW_{s'}^{v,w} + \int_t^s\int_{\mathbb{Z}} \log(1+w_{s'}(z))\left(\nu(ds'\times dz) - \lambda_{s'}^{v,w}(dz)\,ds'\right).\end{aligned} \tag{3.5}$$

for every $t \in \mathbf{T}$ and every $s \in [t,T]$. Substituting (3.3) into (3.2) gives (3.1). □

3.2. **Backward Stochastic Differential-Difference Equation Representation of RDU.** It follows from expression (3.1) for discounted relative entropy that $U_t^{v,w}(C)$ given by (2.7) is rewritten as

$$\begin{aligned}U_t^{v,w}(C) &= E_t^{v,w}\Bigg[\int_t^T e^{-\beta(s-t)}\Bigg\{f(X_s(C), U_s^{v,w}(C)) \\ &\quad + \frac{1}{\zeta}\left(\frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}}\left(\frac{-w_s(z)}{1+w_s(z)} + \log(1+w_s(z))\right)\lambda_s^{v,w}(dz)\right)\Bigg\}\,ds\Bigg],\end{aligned} \tag{3.6}$$

or equivalently

$$\begin{aligned}U_t^{v,w}(C) &= E_t^{v,w}\Bigg[\int_t^T \Bigg\{f^\beta(X_s(C), U_s^{v,w}(C)) \\ &\quad + \frac{1}{\zeta}\left(\frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}}\left(\frac{-w_s(z)}{1+w_s(z)} + \log(1+w_s(z))\right)\lambda_s^{v,w}(dz)\right)\Bigg\}\,ds\Bigg].\end{aligned} \tag{3.7}$$

Let $\mathbf{D}_\lambda^{\exp,\alpha}$ (resp., $\mathbf{D}_\lambda^2$) denote the Banach space of $\mathcal{P}\otimes\mathcal{Z}$-measurable real-valued processes G (resp., H) with norm $E[\int_0^T\int_{\mathbb{Z}} |\exp(\alpha G_t(z)) - 1|^2 \lambda_t(dz)\,dt]$ (resp., $E[\int_0^T\int_{\mathbb{Z}} |H_t(z)|^2\lambda_t(dz)\,dt]$). Let $C \in \mathcal{C}$. Consider the following BSDDE

$$dU_t(C) = -\mu_t^U\,dt + \sigma_t^U\cdot dW_t + \int_{\mathbb{Z}} \Delta U_{t-}(z)\left(\nu(dt\times dz) - \lambda_t(dz)\,dt\right), \quad U_T = 0 \tag{3.8}$$

where $\sigma^U \in \prod_{i=1}^d \mathbf{L}^2$, and $\Delta U \in \mathbf{D}_\lambda^{\exp,-\zeta}$, and

$$\mu_t^U = f^\beta(X_t(C), U_t(C)) - \frac{\zeta}{2}\|\sigma_t^U\|^2 - \int_{\mathbb{Z}}\left(\frac{1}{\zeta}\left(e^{-\zeta\Delta U_{t-}(z)} - 1\right) + \Delta U_{t-}(z)\right)\lambda_t(dz). \tag{3.9}$$

The following proposition presents that if there exists a unique triplet $(U, \sigma^U, \Delta U)$ in $\mathbf{L}^\infty\times\mathbf{L}^2\times\mathbf{D}_\lambda^{\exp,-\zeta}$ to BSDDE (3.8), then $\hat{U}(C)$ is the initial value[5] of U. This extends the result of Skiadas [27] in the case when the underlying utility is non-recursive non-time-additive utility or SDU, and the information filtration is non-Markovian diffusion information.

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[5] To be exact, $U_0(C)$ is a random variable taking some specific value with probability one.

Proposition 2. *Let $C \in \mathcal{C}$. Suppose that there exists a unique triplet $(U, \sigma^U, \Delta U) \in \mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,-\zeta}$ satisfying BSDDE (3.8). Then $U^{v,w}$ satisfies*

$$U_t^{v,w} = U_t + E_t^{v,w}\Big[\int_t^T \Big(f^\beta(X_s, U_s^{v,w}) - f^\beta(X_s, U_s) + \mathcal{Q}_s^{v,w}\Big)\, ds\Big] \quad \forall t \in \mathbf{T} \tag{3.10}$$

where

$$\begin{aligned}\mathcal{Q}_s^{v,w} = \frac{1}{\zeta}\Big\{\frac{1}{2}\|v_s + \zeta\sigma_s^U\|^2 \\ + \int_{\mathbb{Z}}\Big(\frac{-w_s(z)}{1+w_s(z)} + \log(1+w_s(z)) + \frac{e^{-\zeta\Delta U_{s-}(z)} - 1}{1+w_s(z)} + \zeta\Delta U_{s-}(z)\Big)\lambda_s^{v,w}(dz)\Big\}.\end{aligned} \tag{3.11}$$

Furthermore, the minimizer of (2.9) is $P^{\hat{v},\hat{w}}$ where $(\hat{v}_t, \hat{w}_{t-}(z)) = (-\zeta\sigma_t^U, e^{-\zeta\Delta U_{t-}(z)} - 1)$, and $\hat{U}(C) = U_0(C)$.

Proof. Substituting (3.4a) and (3.4b) into (3.8) gives BSDDE

$$\begin{aligned}dU_t = -\Big\{f^\beta(X_t, U_t) - \Big(\frac{\zeta}{2}\|\sigma_t^U\|^2 + \sigma_t^U \cdot v_t\Big) \\ - \int_{\mathbb{Z}}\Big(\frac{1}{\zeta}\,\frac{e^{-\zeta\Delta U_{t-}(z)} - 1}{1+w_t(z)} + \Delta U_{t-}(z)\Big)\lambda_t^{v,w}(dz)\Big\}\, dt \\ + \sigma_t^U \cdot dW_t^{v,w} + \int_{\mathbb{Z}} \Delta U_{t-}(z)\,\big(\nu(dt\times dz) - \lambda_t^{v,w}(dz)\, dt\big), \qquad \tilde{U}_T = 0.\end{aligned} \tag{3.12}$$

It follows from (3.7) and Predictable Representation Property (see p. 239 in Brémaud [3]) that there exists a unique pair $(\sigma^{v,w}, \Delta U^{v,w}) \in \prod_{j=1}^d \mathbf{L}^2 \times \mathbf{D}_\lambda^2$ such that $U^{v,w}$ satisfies BSDDE

$$\begin{aligned}dU_t^{v,w} = -\Big\{f^\beta(X_t, U_t^{v,w}) + \frac{1}{\zeta}\Big(\frac{1}{2}\|v_t\|^2 + \int_{\mathbb{Z}}\Big(\frac{-w_t(z)}{1+w_t(z)} + \log(1+w_t(z))\Big)\lambda_t^{v,w}(dz)\Big)\Big\}\, dt \\ + \sigma_t^{v,w} \cdot dW_t^{v,w} + \int_{\mathbb{Z}} \Delta U_{t-}^{v,w}(z)\,\big(\nu(dt\times dz) - \lambda_t^{v,w}(dz)\, dt\big) \qquad U_T^{v,w} = 0.\end{aligned} \tag{3.13}$$

Combining (3.12) with (3.13) yields

$$\begin{aligned}d(U_t - U_t^{v,w}) = \Big[f^\beta(X_t, U_t^{v,w}) - f^\beta(X_t, U_t) + \frac{1}{\zeta}\Big\{\frac{1}{2}\|v_t + \zeta\sigma_t^U\|^2 \\ + \int_{\mathbb{Z}}\Big(\frac{-w_t(z)}{1+w_t(z)} + \log(1+w_t(z)) + \frac{e^{-\zeta\Delta U_{t-}(z)} - 1}{1+w_t(z)} + \zeta\Delta U_{t-}(z)\Big)\lambda_t^{v,w}(dz)\Big\}\Big] dt + dM_t^{v,w}\end{aligned} \tag{3.14}$$

for every $t \in \mathbf{T}$ where

$$M_t^{v,w} = (\sigma_t^U - \sigma_t^{v,w}) \cdot dW_t^{v,w} + \int_{\mathbb{Z}}\big(\Delta U_t(z) - \Delta U_{t-}^{v,w}(z)\big)\,\{\,\nu(dt\times dz) - \lambda_t^{v,w}(dz)\, dt\}.$$

Equation (3.10) follows from (3.14). It is easy to see that $\mathcal{Q}^{v,w}$ given by (3.11) is nonnegative, and attains zero if and only if $(v_t, w_t(z)) = (\hat{v}_t, \hat{w}_t(z))$. It is obvious that $U_0^{\hat{v},\hat{w}}(C) = U_0(C)$. Since f^β is Lipschitz in utility and $X(C) \in \mathbf{L}^\infty$, the integrand in the right hand side of (3.10) dominates $-K\,|\bar{U}_s - \bar{U}_s^*| + \mathcal{Q}_s^{v,w}$ for some $K \in \mathbb{R}_{++}$, while $\mathcal{Q}_s^{v,w} \geqslant 0$. Therefore, by Lemma C.1, $U_t^{v,w} \geqslant U_t$ $P^{v,w}$-a.s. for every $t \in \mathbf{T}$. $\square$

3.3. **Existence and Uniqueness.** By Proposition 2, in order to show a sufficient condition for the existence and uniqueness of RDU $\hat{U}$, it is enough to present a sufficient condition for the existence and uniqueness of triplet $(U, \sigma^U, \Delta U)$ in $\mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,-\zeta}$ satisfying BSDDE (3.8).

Let $f_\zeta(x, v) = -\zeta f^\beta(x, -\zeta^{-1} v)$, and $C \in \mathcal{C}$. Consider the following BSDDE

$$dV_t = -\mu_t^V \, dt + \sigma_t^V \cdot dW_t + \int_{\mathbb{Z}} \Delta V_{t-}(z) \left(\nu(dt \times dz) - \lambda_t(dz)\, dt\right), \qquad V_T = 0 \quad (3.15)$$

where $\sigma^V \in \prod_{i=1}^d \mathbf{L}^2$, and $\Delta V \in \mathbf{D}_\lambda^{\exp,1}$, and

$$\mu_t^V = f_\zeta(X_t(C), V_t) + \frac{1}{2}\|\sigma_t^V\|^2 + \int_{\mathbb{Z}} \left(e^{\Delta V_{t-}(z)} - 1 - \Delta V_{t-}(z)\right) \lambda_t(dz).$$

It immediately follows from a simple rescaling that if $(U, \sigma^U, \Delta U) \in \mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,-\zeta}$ satisfies BSDDE (3.8) then $(V, \sigma^V, \Delta V) := (-\zeta U, -\zeta\sigma^U, -\zeta\Delta U) \in \mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,1}$ satisfies BSDDE (3.15), and conversely if $(V, \sigma^V, \Delta V) \in \mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,1}$ satisfies (3.15) then $(U, \sigma^U, \Delta U) := (-\zeta^{-1}V, -\zeta^{-1}\sigma^V, -\zeta^{-1}\Delta V) \in \mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,-\zeta}$ satisfies (3.8). It shall be shown that there exists a unique $(V, \sigma^V, \Delta V) \in \mathbf{L}^\infty \times \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,1}$ satisfying BSDDE (3.15). To do so, consider the following recursion

$$V_t = \log\left(E_t\left[\exp\left(\int_t^T f_\zeta(X_s(C), V_s)\, ds\right)\right]\right) \quad \forall t \in \mathbf{T}. \qquad (3.16)$$

The following lemma is a jump-diffusion information version of Lemma A.1 shown by Schroder and Skiadas [25].

Lemma 2. *For any $V \in \mathbf{L}^\infty$, the following statements are equivalent:*

(a) *There exists a unique $(\sigma^V, \Delta V) \in \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,1}$ such that $(V, \sigma^V, \Delta V)$ satisfies (3.15).*
(b) *There exists some $(\sigma^V, \Delta V) \in \mathbf{L}^2 \times \mathbf{D}_\lambda^{\exp,1}$ such that $(V, \sigma^V, \Delta V)$ satisfies (3.15).*
(c) *V satisfies (3.16).*

Proof. See Appendix D.1. □

The following lemma extends Lemma A.3 of Schroder and Skiadas [25] in the case when $f^\beta(x, u) = v(x) - \beta u$ and f_ζ is decreasing in utility, which is not assumed to hold in this paper.

Lemma 3. *Under Assumption 1, there exists a unique solution in $\mathbf{L}^\infty$ to recursion (3.16).*

Proof. Let $C \in \mathcal{C}$. Define a mapping $F : \mathbf{L}^\infty \to \mathbf{L}^\infty$ by

$$F_t(V) = \log\left(E_t\left[\exp\left(\int_t^T f_\zeta(X_s(C), V_s)\, ds\right)\right]\right) \quad \forall t \in \mathbf{T}.$$

It suffices to present that F is well-defined and has a unique fixed point. Schroder and Skiadas [25] show it by utilizing the propoerty that f_ζ is decreasing in utility. This property, however, is not assumed to hold in this paper. Since by Assumption 1, f_ζ satisfies the Lipschitz condition in utility and the generalized growth condition in consumption, and $X(C)$ is in $\mathbf{L}^\infty$, it follows that for each $t \in \mathbf{T}$,

$$\begin{aligned}|f_\zeta(X_s(C), V_s)| &\leqslant \zeta\big|\big(f^\beta(X_s(C), -\zeta^{-1}V_s) - f^\beta(X_s(C), 0)\big)\big| + \big|\zeta f^\beta(X_s(C), 0)\big| \\ &\leqslant \operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} k_1(X_s(C))\,|V_s| + \operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} \zeta\, k_2(X_s(C)) \leqslant K_1\|V\|_{\mathbf{L}^\infty} + K_2\end{aligned}$$

for some $(K_1, K_2) \in \mathbb{R}^2_{++}$, and hence, $F_t(V) \leqslant (K_1\|V\|_{\mathbf{L}^\infty} + K_2)(T-t)$. Thus, $F(V) \in \mathbf{L}^\infty$, and $F : \mathbf{L}^\infty \to \mathbf{L}^\infty$ is well-defined. Let $(V, \tilde{V}) \in \mathbf{L}^\infty \times \mathbf{L}^\infty$. The process $F(V)$ is evaluated as follows:

$$\begin{aligned}F_t(V) &= \log\bigg(E_t\bigg[\exp\bigg(\int_t^T f_\zeta(X_s(C), V_s)\,ds\bigg)\bigg]\bigg) \\ &= \log\bigg(E_t\bigg[\exp\bigg(\int_t^T \Big(f_\zeta(X_s(C), \tilde{V}_s) + f_\zeta(X_s(C), V_s) - f_\zeta(X_s(C), \tilde{V}_s)\Big)\,ds\bigg)\bigg]\bigg) \\ &\leqslant \log\bigg(E_t\bigg[\exp\bigg(\int_t^T \Big(f_\zeta(X_s(C), \tilde{V}_s) + \operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} k_1(X_s(C))\,|V_s - \tilde{V}_s|\Big)\,ds\bigg)\bigg]\bigg) \\ &= F_t(\tilde{V}) + K(T-t)\|V - \tilde{V}\|_{\mathbf{L}^\infty}\end{aligned}$$

where $K = \operatorname{ess\,sup}_{(\omega,s)\in\Omega\times\mathbf{T}} k_1(X_s(C))$. Interchanging the roles of V and $\tilde{V}$ yields $F_t(\tilde{V}) \leqslant F_t(V) + K(T-t)\|V - \tilde{V}\|_{\mathbf{L}^\infty}$. Hence, $|F_t(V) - F_t(\tilde{V})| \leqslant K(T-t)\|V - \tilde{V}\|_{\mathbf{L}^\infty}$. Therefore, it is shown in the same way as the proof of Proposition 1 that F is a contraction mapping, and has a unique fixed point. □

The following theorem follows from Proposition 2 and Lemma 2 and 3, and generalizes the result of Skiadas [27] in the case when the underlying utility is non-recursive non-time-additive utility, and the information filtration is non-Markovian diffusion information.

Theorem 1. *Under Assumption 1, the RDU $\hat{U}$ defined by (2.9) is well-defined, and for every $C \in \mathcal{C}$, $\hat{U}(C) = U_0(C)$ where $U_0(C)$ is the initial value of the unique solution $U_t(C)$ in $\mathbf{L}^\infty$ to BSDDE (3.8).*

4. Normalization of RDU

This section reveals that under Assumption 1, the RDU $\hat{U}$ can be normalized, and exhibits its normalized representation.

Since $\|\sigma_t\|^2$ is the derivative of the quadratic variation $[U]$ of U, let $d[U]_t/dt := \|\sigma_t\|^2$. Then the utility process $U_t(C)$ satisfying BSDDE (3.8) has the following SDU-like representation

$$U_t(C) = E_t\bigg[\int_t^T \bigg\{f^\beta(X_s(C), U_s(C)) - \frac{\zeta}{2}\frac{d[U]_s}{ds} - \int_{\mathbb{Z}}\Big(\frac{1}{\zeta}\big(e^{-\zeta\Delta U_{s-}(z)} - 1\big) + \Delta U_{s-}(z)\Big)\,\lambda_s(dz)\bigg\}\,ds\bigg] \tag{4.1}$$

for every $t \in \mathbf{T}$ where $f^\beta(x, u) = f(x, u) - \beta u$, or equivalently

$$\begin{aligned}U_t(C) = E_t\bigg[\int_t^T e^{-\beta(s-t)}\bigg\{&f(X_s(C), U_s(C)) - \frac{\zeta}{2}\frac{d[U]_s}{ds} \\ &- \int_{\mathbb{Z}}\Big(\frac{1}{\zeta}\big(e^{-\zeta\Delta U_{s-}(z)} - 1\big) + \Delta U_{s-}(z)\Big)\,\lambda_s(dz)\bigg\}\,ds\bigg]\end{aligned} \tag{4.2}$$

for every $t \in \mathbf{T}$. It is easy to see that in each of these representations, the second term is a penalty to the continuous variation of the utility process, and the parameter ζ works as its multiplier. For the third term, the following inequalities hold:

$$\frac{1}{\zeta}\big(e^{-\zeta \Delta U_{s-}(z)} - 1\big) + \Delta U_{s-}(z) \geqslant 0,$$

$$\frac{\partial}{\partial \zeta}\Big(\frac{1}{\zeta}\big(e^{-\zeta \Delta U_{s-}(z)} - 1\big) + \Delta U_{s-}(z)\Big) = \frac{1}{\zeta^2}\Big(1 - (1 + \zeta \Delta U_{s-}(z)) e^{-\zeta \Delta U_{s-}(z)}\Big) \geqslant 0,$$

where equalities hold if and only if $\Delta U_{s-}(z) = 0$. Thus, the third term is a penalty to the jump variation of the utility process, and ζ works as its multiplier. These representations are intuitively comprehensible, but not so analytically intractable. To obtain a more analytically tractable representation of SDU, Duffie and Epstein [9] considers its *ordinally equivalent utility* defined as follows. A utility $\bar{U} : \mathcal{C} \to \mathbb{R}$ is said to be *ordinally equivalent* to a utility $U : \mathcal{C} \to \mathbb{R}$ if and only if there exists a strictly increasing $\mathbf{C}^2$-function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$ such that $\bar{U} = \varphi \circ U$. Then the notion of *normalizable* RDU is defined by the following.

Definition 2. An RDU $U : \mathcal{C} \to \mathbb{R}$ with characteristic (f^β, X, ζ) is *normalizable* if and only if there exists an ordinally equivalent transform φ such that for every $C \in \mathcal{C}$, $\bar{U}_t(C) := \varphi(U_t(C))$ is a solution in $\mathbf{L}^\infty$ to the recursion

$$\bar{U}_t(C) = E_t\left[\int_t^T g(X_s(C), \bar{U}_s(C))\, ds\right] \qquad \forall t \in \mathbf{T} \tag{4.3}$$

where g is given by $g(x, \bar{u}) = \varphi'(\bar{u}) f^\beta(x, \bar{u})$.

The pair (g, X) or the function g is called the *aggregator*. Skiadas [27] derives the normalized robust dynamic utility under diffusion information because any SDU is normalizable under diffusion information (Duffie and Epstein [9]). However, this does not apply to the case of jump-diffusion information. Recently, Kusuda [19] has presented a necessary and sufficient condition for an SDU to be normalizable under jump-diffusion information. For each $C \in \mathcal{C}$, define a utility process by $\bar{U}(C) = \varphi(U(C))$ where φ is an increasing function given by

$$\varphi(u) = \frac{1}{\zeta}(1 - e^{-\zeta u}). \tag{4.4}$$

The following proposition immediately follows from the result of Kusuda [19], and extends the result of Skiadas [27] in the case when the underlying utility is non-recursive non-time-additive utility, *i.e,* $f^\beta(x, u) = v(x) - \beta u$, and the information filtration is non-Markovian diffusion information.

Proposition 3. *Under Assumption 1, any RDU* $U : \mathcal{C} \to \mathbb{R}$ *with characteristic* (f^β, X, ζ) *is normalizable, and the normalized RDU* $\bar{U}$ *has the aggregator* g *of the form*

$$\begin{aligned} g(x, \bar{u}) &= (1 - \zeta \bar{u})\, f^\beta\big(x, -\frac{1}{\zeta}\log(1 - \zeta \bar{u})\big) \\ &= (1 - \zeta \bar{u})\, \Big(f\big(x, -\frac{1}{\zeta}\log(1 - \zeta \bar{u})\big) + \frac{\beta}{\zeta}\log(1 - \zeta \bar{u})\Big), \end{aligned} \tag{4.5}$$

and satisfies $\operatorname{ess\,sup}_{(\omega,t)\in\Omega\times\mathbf{T}} \bar{U}_t(C) < \zeta^{-1}$ *for every* $C \in \mathcal{C}$.

Proof. Note that the following equations hold:

$$\varphi'(u) = e^{-\zeta u} \qquad \forall u, \tag{4.6a}$$

$$\varphi''(u) = -\zeta\, \varphi'(u) \qquad \forall u, \tag{4.6b}$$

$$\varphi'(u) f^{\beta}(x,u) = g(x, \varphi(u)) \qquad \forall (x,u), \tag{4.6c}$$

Applying Ito's Formula to $\bar{U}_t = \varphi(U_t)$ yields BSDDE

$$\begin{aligned} d\bar{U}_t &= -\mu_t^{\bar{U}} dt + \varphi'(U_{t-})\, \sigma_t^U \cdot dW_t \\ &\quad + \int_{\mathbb{Z}} \big(\varphi(U_{t-} + \Delta U_{t-}(z)) - \varphi(U_{t-})\big) \big(\nu(dt \times dz) - \lambda_t(dz)\, dt\big), \quad \bar{U}_T = \varphi(0) = 0 \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \mu_t^{\bar{U}} &= \varphi'(U_{t-}) \Big(\mu_{t-}^U + \int_{\mathbb{Z}} \Delta U_{t-}(z)\, \lambda_t(dz) \Big) - \frac{1}{2} \varphi''(U_{t-})\, \|\sigma_t^U\|^2 \\ &\qquad - \int_{\mathbb{Z}} \big(\varphi(U_{t-} + \Delta U_{t-}(z)) - \varphi(U_{t-})\big) \lambda_t(dz). \end{aligned} \tag{4.8}$$

It is confirmed that $\varphi'(U_{t-})\, \sigma_t^U \in \prod_{j=1}^d \mathbf{L}^2$ and $(\varphi(U_{t-} + \Delta U_{t-}(z)) - \varphi(U_{t-})) \in \mathbf{D}_\lambda^2$. Substituting (3.9) into (4.8) gives

$$\begin{aligned} \mu_t^{\bar{U}} &= \varphi'(U_{t-}) f^{\beta}(X_{t-}, U_{t-}) - \frac{\zeta}{2} \varphi'(U_{t-})\, \|\sigma_t^U\|^2 \\ &\quad - \varphi'(U_{t-}) \int_{\mathbb{Z}} \Big(\frac{1}{\zeta} (e^{-\zeta \Delta U_{t-}} - 1) - \Delta U_{t-}(z) \Big) \lambda_t(dz) \\ &\quad - \frac{1}{2} \varphi''(U_{t-})\, \|\sigma_t^U\|^2 - \int_{\mathbb{Z}} \Big(\varphi(U_{t-} + \Delta U_{t-}(z)) - \varphi(U_{t-}) - \varphi'(U_t)\, \Delta U_{t-}(z) \Big) \lambda_t(dz) \\ &= \varphi'(U_{t-}) f^{\beta}(X_{t-}, U_{t-}) - \frac{1}{2} \big(\zeta\, \varphi'(U_{t-}) + \varphi''(U_{t-}) \big) \|\sigma_t^U\|^2 \\ &\quad + \int_{\mathbb{Z}} \int_{U_{t-}}^{U_{t-} + \Delta U_{t-}(z)} \Big(\varphi'(U_{t-}) e^{-\zeta(u - U_{t-})} - \varphi'(u) \Big) du\, \lambda_t(dz). \end{aligned} \tag{4.9}$$

Substituting (4.6a)-(4.6c) into (4.9) yields $\mu_t^{\bar{U}} = g(X_{t-}, \bar{U}_{t-})$, and therefore the normalized representation (4.3). Since $U_t(C) \in \mathbf{L}^\infty$ and $\bar{U}_t(C) = \zeta^{-1}(1 - e^{-\zeta U_t(C)})$, $\operatorname{esssup}_{(\omega,t) \in \Omega \times \mathbf{T}} \bar{U}_t(C) < \zeta^{-1}$. □

Example 5. Consider the RDU such that the underlying dynamic utility is non-recursive habit formation utility, *i.e*, $C_t = \int_0^t c_s ds$, $X_t(C) = (c_t, x_t)$, and $f^{\beta}(c,x,u) = v(c - x) - \beta u$. Then the aggregator of the corresponding robust utility is of the form

$$g(c, x, \bar{u}) = (1 - \zeta \bar{u}) \Big(v(c - x) + \frac{\beta}{\zeta} \log(1 - \zeta \bar{u}) \Big).$$

In particular, if there is no habit-forming effect and $v(c) = \log(c)$ then the RDU is a Kreps-Porteus utility (Kreps and Porteus [18]) studied by Duffie and Epstein [9] and Schroder and Skiadas [25], although in the Kreps-Porteus utility, ζ is a measure of comparative risk aversion.

Example 6. Consider the RDU such that the underlying dynamic utility is the Kreps-Porteus habit formation whose aggregator is of the form

$$f^\beta(c,x,u) = \begin{cases} (1-\alpha)\left(\dfrac{(c-x)^\gamma}{\gamma}|u|^{\frac{\alpha}{\alpha-1}} - \beta u\right) & \text{if } \gamma \neq 0, \\ \\ (1-\alpha u)\left(\log(c-x) + \dfrac{\beta}{\alpha}\log(1-\alpha u)\right) & \text{if } \gamma = 0. \end{cases}$$

Then the aggregator of the corresponding robust utility is of the form

$$\begin{aligned} & g(c,x,\bar{u}) \\ &= \begin{cases} (1-\alpha)\,(1-\zeta\bar{u})\left(\dfrac{(c-x)^\gamma}{\gamma}\left|\dfrac{1}{\zeta}\log(1-\zeta\bar{u})\right|^{\frac{\alpha}{\alpha-1}} + \dfrac{\beta}{\zeta}\log(1-\zeta\bar{u})\right) & \text{if } \gamma \neq 0, \\ \\ (1-\zeta\bar{u})\left(1+\dfrac{\alpha}{\zeta}\log(1-\zeta\bar{u})\right)\left(\log(c-x) + \dfrac{\beta}{\alpha}\log\Big(1+\dfrac{\alpha}{\zeta}\log(1-\zeta\bar{u})\Big)\right) & \text{if } \gamma = 0. \end{cases} \end{aligned}$$

5. Basic Properties

This section exmines the RDU's basic properties including time consistency, preferences for information, monotonicity, and "ambiguity aversions." Time consistency and monotonicity are presented by generalizing results of Duffie and Epstein [9] for normalized SDUs. Preferences for information are analyzed by exploiting the result of Skiadas [26]. "Ambiguity aversions" are examined by mainly adopting the notions advocated by Epstein [11], Epstein and Zhang [13], and Chen and Epstein [5].

5.1. **Time Consistency and Preferences for Information.** Duffie and Epstein [9] show that any SDU with aggregator satisfying the uniform Lipschitz condition is *time consistent* (for definition, see Duffie and Epstein [9]). Skiadas [26] presents that if an SDU aggregator with the uniform Lipschitz condition is convex (resp., concave) in utility, the SDU is *information seeking* (resp., *information averse*) (for definitions, see Skiadas [26]). It is easily found that in their proofs, the uniform Lipschitz condition can be replaced with the *quasi-Lipschitz condition* defined in the following lemma.

Lemma 4. *Under Assumption 1, the aggregator g defined by (4.5) satisfies the generalized growth condition in consumption and the following condition:*

Quasi-Lipschitz condition in utility: *Let $C \in \mathcal{C}$ and $(U, \tilde{U}) \in \mathbf{L}^\infty \times \mathbf{L}^\infty$ such that* $\operatorname{ess\,sup}_{(\omega,t)\in\Omega\times\mathbf{T}} \max\{U_t, \tilde{U}_t\} < \zeta^{-1}$. *There exists $K \in \mathbb{R}_{++}$ such that*

$$|g(X_t(C), U_t) - g(X_t(C), \tilde{U}_t)| \leqslant K|U_t - \tilde{U}_t| \qquad \forall(\omega,t) \in \Omega \times \mathbf{T}.$$

Proof. See Appendix D.2. □

Proposition 4. *Any RDU $\hat{U}$ with characteristic (f^β, X, ζ) satisfying Assumption 1 is time consistent. In addition, if $(-\zeta f^\beta_u + f^\beta_{uu})$ is non-negative (resp., non-positive), then $\hat{U}$ is information seeking (resp., information averse), where f^β_u and f^β_{uu} are the first and the second partial derivatives of f^β with respect to utility, respectively. In particular, if the underlying utility is a non-recursive utility, i.e., $f^\beta(\omega,t,x,u) = v(x) - \beta u$, then $\hat{U}$ is information seeking.*

Proof. Time consistency follows from Proposition 4 in Duffie and Epstein [9] and Lemma 4. For proof of preferences for information, by Proposition A in Skiadas [26] and Lemma 4, it is engouh to show the claim that the sign of $g_{\bar{u}\bar{u}}$ coincides with that of $(-\zeta f^\beta_u + f^\beta_{uu})$. It follows that $g_{\bar{u}} = -\zeta f^\beta + f^\beta_u$, and then $g_{\bar{u}\bar{u}} = (1 - \zeta\bar{u})(-\zeta f^\beta_u + f^\beta_{uu})$. The claim was, therefore, shown. Let $f^\beta(\omega, t, x, u) = v(x) - \beta u$. Then $-\zeta f^\beta_u + f^\beta_{uu} = \beta\zeta \geq 0$, and then $\hat{U}$ is information seeking. □

5.2. **Monotonicity.** A characteristic (f^β, X, ζ) or an aggregator (g, X) is said to be *regular* if and only if Assumption 1 and the following condition holds:

- (f^β, X) *is increasing in consumption process*, which means that for every $(C, \tilde{C}) \in \mathcal{C} \times \mathcal{C}$ with $C \geqslant \tilde{C}$, $f^\beta(X_t(C), u) \geqslant f^\beta(X_t(\tilde{C}), u)$ for every $(t, u) \in \mathbf{T} \times \mathbb{R}$, P-a.s.

Note that if (f^β, X) is increasing in consumption process then so is (g, X).

Proposition 5. *Any RDU with regular aggregator is increasing.*

Proof. Let $\bar{U}$ be an RDU with regular aggregator (g, X). Let $(C, \tilde{C}) \in \mathcal{C} \times \mathcal{C}$ with $C \geqslant \tilde{C}$. Let $\bar{U}^C := \bar{U}(C)$ and $\bar{U}^{\tilde{C}} := \bar{U}(\tilde{C})$. It follows that for every $t \in \mathbf{T}$,

$$\bar{U}^C_t - \bar{U}^{\tilde{C}}_t = E_t\left[\int_t^T \Big(g(X^C_s, \bar{U}^C_s) - g(X^{\tilde{C}}_s, \bar{U}^{\tilde{C}}_s)\Big)\, ds\right],$$

and since g is quasi-Lipschitz in utility and (g, X) is increasing in consumption process, the integrand of the above equation is evaluated as

$$g(X^C_s, \bar{U}^C_s) - g(X^{\tilde{C}}_s, \bar{U}^{\tilde{C}}_s) \geqslant g(X^C_s, \bar{U}^C_s) - g(X^{\tilde{C}}_s, \bar{U}^C_s) - K\,|\bar{U}^C_s - \bar{U}^{\tilde{C}}_s| \geqslant -K\,|\bar{U}^C_s - \bar{U}^{\tilde{C}}_s|$$

for some $K \in \mathbb{R}_{++}$. The result follows by Lemma C.1. □

5.3. **Comparative Ambiguity Aversion.** Through this subsection, agents are assumed to have the common reference probability P.

A characteristic (f^β, X, ζ) or an aggregator (g, X) is said to be *event-independent* if and only if the following conditions hold:

(a) f^β does not depend on $\omega \in \Omega$.
(b) X does not depend on $\omega \in \Omega$ and $X(C)$ is a deterministic process for every deterministic consumption process $C \in \mathcal{C}$.

The following lemma is useful for comparing two utilities.

Lemma 5. *Let $C \in \mathcal{C}$. Suppose that a utility process $U_t(C)$ satisfies BSDDE (3.8), and that $U^* := U^*(C)$ satisfies BSDDE*

$$dU^*_t = -\mu^{U^*}_t dt + \sigma^{U^*}_t \cdot dW_t + \int_{\mathbb{Z}} \Delta U^*_{t-}(z)\,\{\,\nu(dt \times dz) - \lambda_t(dz)\,dt\,\}, \qquad U^*_T = 0$$

where $\sigma^{U^*} \in \prod_{j=1}^d \mathbf{L}^2$, $\Delta U^* \in \mathbf{D}^{\exp,-\zeta^*}_\lambda$, and

$$\mu^{U^*}_t = f^\beta(X_t, U^*_t) - \frac{\zeta^*}{2}\,\|\sigma^{U^*}_t\|^2 - \int_{\mathbb{Z}} \int_0^{\Delta U^*_{t-}(z)} \Big(1 - e^{-\zeta^* u}\Big)\, du\, \lambda_t(dz).$$

Let φ be the ordinally equivalent transform defined by (4.4) and $\bar{U}^* := \varphi(U^*)$. Then $\bar{U}^*$ satisfies

$$\bar{U}^*_t = E_t\left[\int_t^T \Big\{g(X_s, \bar{U}^*_s) - Z_s(U^*, \zeta^*, \zeta)\Big\}\, ds\right] \tag{5.1}$$

where

$$Z_s(U^*,\zeta^*,\zeta) = \frac{1}{2}(\zeta^*-\zeta)\,\varphi'(U^*_s)\,\|\sigma_s^{\bar{U}^*}\|^2 \\ + \varphi'(U^*_{s-})\int_{\mathbb{Z}}\int_0^{\Delta U^*_{s-}(z)}\Big(1-e^{-(\zeta^*-\zeta)u}\Big)du\,\lambda_s(dz). \quad (5.2)$$

Furthermore, $Z(U^*,\zeta^*,\zeta)\geqslant 0$.

Proof. Applying Ito's Formula to $\bar{U}^* := \varphi(U^*)$ yields (5.1) and (5.2). It is easy to see that $Z(U^*,\zeta^*,\zeta)\geqslant 0$. □

Chen and Epstein [5] define the notion of *comparative ambiguity aversion* in the following way (for formal arguments, see Epstein [11] and Epstein and Zhang [13]). An event $A\in\mathcal{F}_T$ is said to be *unambiguous* if and only if $\tilde{P}(A)=P(A)$ for every $\tilde{P}\in\mathbb{P}$. Let $\bar{\mathcal{A}}$ denote the class of unambiguous events. Let $\bar{\mathcal{A}}_t := \bar{\mathcal{A}}\cap\mathcal{F}_t$ for every $t\in\mathbf{T}$. A consumption process C is said to be *unambiguous* if and only if C_t is $\bar{\mathcal{A}}_t$-measurable for every $t\in\mathbf{T}$. Let $\mathcal{C}_{\bar{\mathcal{A}}}$ denote the set of all unambiguous consumption processes. The notion of *comparative ambiguity aversion* is defined in the following.

Definition 3. Let $\hat{U}$ and $\hat{U}^*$ be RDUs with corresponding classes $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}^*$ of unambiguous events. It is said that $\hat{U}^*$ is more ambiguity averse than $\hat{U}$ if and only if $\bar{\mathcal{A}}\supset\bar{\mathcal{A}}^*$ and for every consumption process $C\in\mathcal{C}$ and every $\bar{\mathcal{A}}^*$-unambiguous consumption process $C^*\in\mathcal{C}_{\bar{\mathcal{A}}^*}$, the following holds:

$$\hat{U}(C)\leqslant\hat{U}(C^{\bar{\mathcal{A}}^*}) \quad\Longrightarrow\quad \hat{U}^*(C)\leqslant\hat{U}^*(C^{\bar{\mathcal{A}}^*}).$$

The condition $\bar{\mathcal{A}}\supset\bar{\mathcal{A}}^*$ in the definition of comparative ambiguity aversion, means that the more ambiguous averse agent views more events as ambiguous.

It must be noted that in this paper's setting, clearly an event is unambiguous if and only if the event is deterministic. Therefore, the set $\bar{\mathcal{A}}$ of unambiguous events coincides with the set of deterministic events, and the set $\mathcal{C}_{\bar{\mathcal{A}}}$ of unambiguous consumption processes coincides with the set of deterministic processes.[6] The following lemma shows that in the case when the consumption set is restricted to the set of unambiguous consumption processes (or equivalently, for the set of deterministic consumption processes), any RDU with regular event-independent characteristic is the corresponding dynamic utility.

Lemma 6. *Let* $\hat{U}$ *be an RDU with regular event-independent characteristic* (f^β,X,ζ). *For any* $C^{\bar{\mathcal{A}}}\in\mathcal{C}_{\bar{\mathcal{A}}}$, $\hat{U}(C^{\bar{\mathcal{A}}})=U_0(C^{\bar{\mathcal{A}}})$ *where* $U_0(C^{\bar{\mathcal{A}}})$ *is a unique solutiton in* $\mathbf{L}^\infty$ *to the recursion*

$$U_t(C^{\bar{\mathcal{A}}}) = \int_t^T f^\beta(X_s(C^{\bar{\mathcal{A}}}),U_s(C^{\bar{\mathcal{A}}}))\,ds \qquad \forall t\in\mathbf{T}. \quad (5.3)$$

Proof. See Appendix D.3. □

Consider an agent with an RDU whose characteristic (f,X,β,ζ) is regular and event-independent. The following proposition demonstrates that the larger the parameter ζ is, the more ambiguous averse the agent is.

[6] Chen and Epstein [5] present the *κ-ignorance multiple-priors utility* in which the set of the deterministic events is a proper subset of unambiguous events (for details, see Chen and Epstein [5]). The RDU also can be generalized in the same way, but the generalized RDU would be no longer normalizable.

Proposition 6. *Let $\hat{U}$ and $\hat{U}^*$ be RDUs with corresponding characteristics (f, X, β, ζ) and $(f^*, X^*, \beta^*, \zeta^*)$ such that (f, X, β, ζ) is regular and event-independent. Suppose that $(f^*, X^*, \beta^*) = (f, X, \beta)$. Then if $\zeta^* \geqslant \zeta$, then $\hat{U}^*$ is more ambiguity averse than $\hat{U}$.*

Proof. It is obvious that $\bar{\mathcal{A}} = \bar{\mathcal{A}}^*$. By Lemma 6, $\hat{U}_t(C^{\bar{\mathcal{A}}^*}) = \hat{U}_t^*(C^{\bar{\mathcal{A}}^*})$ for every $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$. Thus, $\hat{U}^*$ is more ambiguity averse than $\hat{U}$ if and only if for every $C \in \mathcal{C}$, $\hat{U}(C) \geqslant \hat{U}^*(C)$. Let $C \in \mathcal{C}$. Let φ be the ordinally equivalent transform defined by (4.4), and let $\bar{U} = \varphi(\hat{U})$ and $\bar{U}^* = \varphi(\hat{U}^*)$. Assume $\zeta^* > \zeta$. It is sufficient to show that $\bar{U}(C) \geqslant \bar{U}^*(C)$. The transform φ normalizes $\hat{U}$ to the normalized representation $\bar{U}$ of the form (4.3). Since $\zeta^* > \zeta$, it follows from Lemma 5 that $\bar{U}^* := \bar{U}^*(C)$ satisfies

$$\bar{U}_t^* = E_t\left[\int_t^T \left\{ g(X_s, \bar{U}_s^*) - Z_s(\hat{U}^*, \zeta^*, \zeta) \right\} ds\right] \qquad t \in \mathbf{T} \tag{5.4}$$

where $Z(\hat{U}^*, \zeta^*, \zeta)$ is given by (5.2). Subtracting (4.3) from (5.4) yields

$$\bar{U}_t - \bar{U}_t^* = E_t\left[\int_t^T \left\{ g(X_s, \bar{U}_s) - g(X_s, \bar{U}_s^*) + Z_s(\hat{U}^*, \zeta^*, \zeta) \right\} ds\right] \quad \forall t \in \mathbf{T}. \tag{5.5}$$

Since g is quasi-Lipschitz in utility, the integrand dominates $-K\,|\bar{U}_s - \bar{U}_s^*| + Z_s(\hat{U}^*, \zeta^*, \zeta)$ for some $K \in \mathbb{R}_{++}$, while $Z_s(\hat{U}^*, \zeta^*, \zeta) \geqslant 0$. Then it follows from Lemma C.1 that $\bar{U}(C) \geqslant \bar{U}^*(C)$. $\square$

5.4. **Absolute Ambiguity Aversion.** In order to define the absolute ambiguity aversion of a utility, Chen and Epstein [5] introduced the notion of *probabilistically sophisticated utility for timeless prospects* (for definition, see Chen and Epstein [5]) as "ambiguity neutral" utility because the well known notion of "probabilistically sophisticated utility" introduced by Machina and Schmeidler [23], is not appropriate in a dynamic setting (for details, see Chen and Epstein [5]). In this paper, the following notion is introduced.

Definition 4. The *ambiguity neutral version of an RDU with characteristic* (f^β, X, ζ) is the dynamic utility with characteristic (f^β, X).

It is easily conjectured that an RDU with characteristic (f^β, X, ζ) is probabilistically sophisticated utility for timeless prospects if and only if the characteristic is $(f^\beta, X, 0+)$, but it is beyond the scope of this paper to explore the conjecture.

The meaning of the following definition is intuitively clear, which states that an RDU is ambiguity averse if and only if whenever its ambiguity neutral version rejects an ambiguous consumption plan against an unambiguous consumption plan, the RDU rejects the ambiguous consumption plan against the unambiguous consumption plan.

Definition 5. An RDU is *ambiguity averse* if and only if the RDU is more ambiguity averse than its ambiguity neutral version.

It immediately follows from Proposition 6 that any RDU with regular event-independent characteristic is ambiguity averse.

Proposition 7. *Any RDU with regular event-independent characteristic is ambiguity averse.*

Appendix A. Marked Point Process

A.1. **Definitions.** A double sequence $(s_n, Z_n)_{n\in\mathbb{N}}$ is considered, where s_n is the occurrence time of an nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define a random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n\in\mathbb{N}} 1_{\{s_n \leqslant t,\, Z_n \in A\}} \qquad \forall (t, A) \in \mathbf{T} \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$-*marked point process*.

Let λ be such that

(a) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\lambda_t(\omega, \cdot\,)$ is a finite Borel measure on $\mathbb{Z}$.
(b) For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

The marked point process $\nu(dt\times dz)$ is said to have the P-*intensity kernel* $\lambda_t(dz)$ if and only if the equation $E\left[\int_0^T Y_s\, \nu(ds \times A)\right] = E\left[\int_0^T Y_s \lambda_s(A)\, ds\right]$ holds for every $A \in \mathcal{Z}$ and for every nonnegative $\mathcal{P}$-measurable process Y.

A.2. **Integration Theorem.** Let $\nu(dt\times dz)$ be a $\mathbb{Z}$-marked point process with the P-intensity kernel $\lambda_t(dz)$. Let H be a $\mathcal{P}\otimes\mathcal{Z}$-measurable function. It follows that:

(a) If the integrability condition $E\left[\int_0^T\int_{\mathbb{Z}} |H_s(z)|\lambda_s(dz)\, ds\right] < \infty$ holds, then the process $\int_0^t\int_{\mathbb{Z}} H_s(z)\big(\nu(ds\times dz) - \lambda_s(dz)\, ds\big)$ is a P-martingale.
(b) If $H \in \mathcal{L}^1(\lambda_t(dz)\times dt)$, then the process $\int_0^t\int_{\mathbb{Z}} H_s(z)\big(\nu(ds\times dz) - \lambda_s(dz)\, ds\big)$ is a local P-martingale.

Proof. See p. 235 in Brémaud [3]. □

Appendix B. Ito's Formula and Girsanov's Theorem

B.1. **Ito's Formula.** Let X be a n-dimensional semimartingale. Then for any $\mathbf{C}^{1,2}$-function $g : \mathbf{T}\times\mathbb{R}^n \to \mathbb{R}$, $g(t, X)$ is a semimartingale of the form

$$\begin{aligned} g(t, X_t) &= g(0, X_0) + \int_0^t \frac{\partial}{\partial s} g(s, X_s)\, ds \\ &+ \sum_{j=1}^n \int_0^t \frac{\partial}{\partial x_j} g(s-, X_{s-})\, dX_s^j + \frac{1}{2}\sum_{j=1}^n\sum_{k=1}^n \int_0^t \frac{\partial^2}{\partial x_j \partial x_k} g(s-, X_{s-})\, d\langle X^{jc}, X^{kc}\rangle \\ &\qquad + \sum_{0\leqslant s\leqslant t}\left\{ g(s-, X_s) - g(s-, X_{s-}) - \sum_{j=1}^d \frac{\partial}{\partial x_j} g(s-, X_{s-})\, \Delta X_s^j \right\} \end{aligned}$$

where X^{jc} is the continuous part of X^j and $\langle X^{jc}, X^{kc}\rangle$ is quadratic covariation of X^{jc} and X^{kc}.

B.2. **Girsanov's Theorem.** Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ the complete filtered probability space given in this paper.

(a) Let $v \in \prod_{j=1}^d \mathcal{L}^2$ and $w \in \mathcal{L}^1(\lambda_t(dz)\times dt)$ such that $w_t(z) > -1$ P-a.s. for every $(t, z) \in \mathbf{T}\times\mathbb{Z}$. Define a process Λ by

$$\frac{d\Lambda_t}{\Lambda_{t-}} = v_t \cdot dW_t + \int_{\mathbb{Z}} w_t(z)\,\big(\nu(dt\times dz) - \lambda_t(dz)\, dt\big) \qquad \forall t \in \mathbf{T}$$

with $\Lambda_0 = 1$, and suppose $E[\Lambda_T] = 1$. Then there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ given by the Radon-Nikodym derivative

$$d\tilde{P} = \Lambda_T \, dP$$

such that:

(i) The measure $\tilde{P}$ is equivalent to P.

(ii) The process given by $\tilde{W}_t = W_t - \int_0^t v_s \, ds$ for every $t \in \mathbf{T}$, is a $\tilde{P}$-Wiener process.

(iii) The marked point process $\nu(dt \times dz)$ has the $\tilde{P}$-intensity kernel such that $\tilde{\lambda}_t(dz) = (1 + w_t(z))\lambda_t(dz)$ for every $(t, z) \in \mathbf{T} \times \mathbb{Z}$.

(b) Every probability measure equivalent to P has the structure above.

Appendix C. Extension of Gronwall-Bellman Inequality

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability whose filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbf{T}}$ satisfies the usual conditions. The following lemma is an extension of Gronwall-Bellman Inequality, shown by Skiadas in Duffie and Epstein [9].

Lemma C.1. *Suppose that Y and Z are integrable optional processes, and $K \in \mathbb{R}_{++}$. Suppose that the map defined by $s \mapsto Y_s$ is right continuous, and that the map defined by $s \mapsto E_t[Y_s]$ is continuous P-a.s. for every $t \in \mathbf{T}$. If $Y_T \geqslant 0$ P-a.s., and for every $t \in \mathbf{T}$, $Y_t = E_t[\int_t^T Z_s \, ds + Y_T]$ P-a.s., and $Z_t \geqslant -K|Y_t|$ P-a.s., then $Y_t \geqslant 0$ P-a.s. for every $t \in \mathbf{T}$.*

Proof. See Duffie and Epstein [9]. □

Appendix D. Proofs

D.1. **Proof of Lemma 2.** The proof of Schroder and Skiadas [25] for diffusion information, is extended to jump-diffusion information.

(c) ⇒ (b). Suppose that (c) holds. Define the process Y by $Y_t = \exp(V_t)$. Then the following holds:

$$Y_t = \exp\left(-\int_0^t f_\zeta(X_s(C), V_s) \, ds\right) M_t \qquad \forall t \in \mathbf{T} \tag{D.1}$$

where M is the martingale given by

$$M_t = E_t\left[\exp\left(\int_0^T f_\zeta(X_s(C), V_s) \, ds\right)\right]. \tag{D.2}$$

By Predictable Representation Property, there exists a unique pair $(\sigma^M, \Delta M) \in \mathbf{L}^2 \times \mathbf{D}^2_\lambda$ such that

$$dM_t = \sigma_t^M \cdot dW_t + \int_{\mathbb{Z}} \Delta M_{t-}(z) \left(\nu(dt \times dz) - \lambda_t(dz) \, dt\right).$$

Define processes v^Y and w^Y by $v_t^Y = M_t^{-1} \sigma_t^M$ and $w_t^Y(z) = M_t^{-1} \Delta M_t(z)$, respectively. It follows from (D.1) that $\log M_t = V_t + \int_0^t f_\zeta(X_s(C), V_s) \, ds$. Since V_t and $f_\zeta(X_t(C), V_t)$ are in $\mathbf{L}^\infty$, $\log M \in \mathbf{L}^\infty$, and thus, $(v^Y, w^Y) \in \mathbf{L}^2 \times \mathbf{D}^2_\lambda$. Applying integration by parts to Y_t yields

$$\frac{dY_t}{Y_{t-}} = -f_\zeta(X_t(C), V_t) \, dt + v_t^Y \cdot dW_t + \int_{\mathbb{Z}} w_{t-}^Y(z) \left(\nu(dt \times dz) - \lambda_t(dz) \, dt\right). \tag{D.3}$$

Let $(\sigma^V, \Delta V) = (v^Y, \log(1 + w^Y))$. Since $(v^Y, w^Y) \in \mathbf{L}^2 \times \mathbf{D}^2_\lambda$, $(\sigma^V, \Delta V) \in \mathbf{L}^2 \times \mathbf{D}^{\exp,1}_\lambda$. Applying Ito's Formula to $V_t = \log(Y_t)$ gives (3.15).

(b) $\Rightarrow$ (c). Suppose that $(V, \sigma^V, \Delta V)$ satisfies (3.15), and define Y by $Y_t = \exp(V_t)$. By Ito's Formula, Y satisfies (D.3). Let M be the stochastic exponential of $(\sigma^V, e^{\Delta V} - 1)$, *i.e.*, M_t is the unique local martingale satisfying

$$\frac{dM_t}{M_{t-}} = \sigma^V_t \cdot dW_t + \int_{\mathbb{Z}} \left(e^{\Delta V_{t-}(z)} - 1\right) \left(\nu(dt \times dz) - \lambda_t(dz)\, dt\right) \qquad \forall t \in \mathbf{T},$$

and $M_0 = 1$. Then, M is a martingale. Applying integration by parts to M_t yields (D.1), and (3.16) then follows from (D.1).

(a) $\Leftrightarrow$ (b). Clearly, (a) implies (b). Suppose that (b) holds, and let $Y_t = \exp(V_t)$. Then σ^V and ΔV represent the diffusion term and the jump term of Ito Decomposition of dY_t/Y_{t-}, respectively, and is uniquely determined in $\mathbf{L}^\infty \times \mathbf{D}^{\exp,1}_\lambda$.

D.2. **Proof of Lemma 4.** First, since $g(x, 0) = f^\beta(x, 0)$, g satisfies the generalized growth condition. Let $C \in \mathcal{C}$ and $(U, \tilde{U}) \in \mathbf{L}^\infty \times \mathbf{L}^\infty$ such that $\operatorname{ess\,sup}_{(\omega,t)\in\Omega\times\mathbf{T}} \max\{U_t, \tilde{U}_t\} < \zeta^{-1}$. Let $X := X(C)$. The following holds:

$$\begin{aligned} &g(X_t, U_t) - g(X_t, \tilde{U}_t) \\ &\quad = (1 - \zeta U_t)\big(f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta U_t)\big) - f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta\tilde{U}_t)\big)\big) \\ &\qquad\qquad + \zeta f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta\tilde{U}_t)\big)(U_t - \tilde{U}_t), \end{aligned} \tag{D.4}$$

or

$$\begin{aligned} &g(X_t, U_t) - g(X_t, \tilde{U}_t) \\ &\quad = (1 - \zeta \tilde{U}_t)\big(f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta U_t)\big) - f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta\tilde{U}_t)\big)\big) \\ &\qquad\qquad + \zeta f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta U_t)\big)(\tilde{U}_t - U_t). \end{aligned} \tag{D.5}$$

It follows from Assumption 1 and $0 \leqslant \log(1 + x) \leqslant x$ for every $x \in [0, \infty)$ that the term $f^\beta(X_t, -\zeta^{-1}\log(1 - \zeta U_t)) - f^\beta(X_t, -\zeta^{-1}\log(1 - \zeta\tilde{U}_t))$ is evaluated as

$$\begin{aligned} &\left|f^\beta\ \big(X_t, -\zeta^{-1}\log(1 - \zeta U_t)\big) - f^\beta\big(X_t, -\zeta^{-1}\log(1 - \zeta\tilde{U}_t)\big)\right| \\ &\qquad \leqslant k_1(X_t)\left|-\zeta^{-1}\log\Big(\frac{1 - \zeta U_t}{1 - \zeta\tilde{U}_t}\Big)\right| \leqslant \begin{cases} \dfrac{k_1(X_t)}{1 - \zeta U_t}(\tilde{U}_t - U_t) & \text{if } U_t \leqslant \tilde{U}_t \\[2ex] \dfrac{k_1(X_t)}{1 - \zeta \tilde{U}_t}(U_t - \tilde{U}_t) & \text{if } U_t > \tilde{U}_t \end{cases} \end{aligned} \tag{D.6}$$

It follows by Assumption 1 and $\operatorname{ess\,sup}_{(\omega,t)\in\Omega\times\mathbf{T}} \max\{\tilde{U}_t, U_t\} < \zeta^{-1}$ that $f^\beta(X_t, -\zeta^{-1}\log(1 - \zeta\tilde{U}_t))$ and $f^\beta(X_t, -\zeta^{-1}\log(1 - \zeta U_t))$ are bounded. Therefore, it follows from (D.4), (D.5), and (D.6) that the quasi-Lipschitz condition holds.

D.3. **Proof of Lemma 6.** Since (f^β, X, ζ) is event-independent, it follows from (5.3) that $U_t := U_t(C^{\bar{\mathcal{A}}})$ satisfies the backward ordinary differential equation

$$dU_t = -f^\beta(X_t, U_t)\, dt, \qquad U_T = 0. \tag{D.7}$$

It follows from (3.6) and Predictable Representation Property that there exists a unique pair $(\sigma^{v,w}, \Delta U^{v,w}) \in \mathbf{L}^2 \times \mathbf{D}^2_\lambda$ such that $U^{v,w} := U^{v,w}(C^{\bar{\mathcal{A}}})$ satisfies

BSDDE (3.13). Combining (D.7) with (3.13) gives

$$d(U_t - U_t^{v,w}) = \Bigg[f^\beta(X_t, U_t^{v,w}) - f^\beta(X_t, U_t) + \frac{1}{\zeta}\Bigg\{ \frac{1}{2}\|v_t\|^2 + \int_{\mathbb{Z}} \left(\frac{-w_t(z)}{1+w_t(z)} + \log(1+w_t(z)) \right) \lambda_t^{v,w}(dz) \Bigg\} \Bigg] dt + dM_t^{v,w} \tag{D.8}$$

where

$$M_t^{v,w} = -\sigma_t^{v,w} \cdot dW_t^{v,w} - \int_{\mathbb{Z}} \Delta U_{t-}^{v,w}(z) \{ \nu(dt \times dz) - \lambda_t^{v,w}(dz)\, dt \}.$$

Thus, $U^{v,w}$ satisfies

$$U_t^{v,w} = U_t + E_t^{v,w}\left[\int_t^T \{ f^\beta(X_s, U_s^{v,w}) - f^\beta(X_s, U_s) + \mathcal{Q}_s^{v,w} \}\, ds \right] \quad \forall t \in \mathbf{T}$$

where

$$\mathcal{Q}_s^{v,w} = \frac{1}{\zeta}\left\{ \frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}} \left(\frac{-w_s(z)}{1+w_s(z)} + \log(1+w_s(z)) \right) \lambda_s^{v,w}(dz) \right\}.$$

Since f^β is Lipschitz in utility and $X(C) \in \mathbf{L}_\infty$, the integrand dominates $-K\,|\bar{U}_s - \bar{U}_s^*| + \mathcal{Q}_s^{v,w}$ for some $K \in \mathbb{R}_{++}$, while $\mathcal{Q}_s^{v,w} \geqslant 0$. Thus, by Lemma C.1, $U_t \leqslant U_t^{v,w}$ a.s. for every $t \in \mathbf{T}$, and therefore, $\hat{U}(C^{\bar{\mathcal{A}}}) = U_0(C^{\bar{\mathcal{A}}})$.

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